

On hypersemigroups

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Abstract. We prove that a nonempty subset B of a regular hypersemigroup H is a bi-ideal of H if and only if it is represented in the form $B = A * C$ where A is a right ideal and C a left ideal of H . We also show that an hypersemigroup H is regular if and only if the right and the left ideals of H are idempotent, and for every right ideal A and every left ideal B of H , the product $A * B$ is a quasi-ideal of H . Our aim is not just to add a publication on hypersemigroups but, mainly, to publish a paper which serves as an example to show what an hypersemigroup is and give the right information concerning this structure. We never work directly on an hypersemigroup. If we want to get a result on an hypersemigroup, then we have to prove it first for a semigroup and transfer its proof to hypersemigroup. But there is further interesting information concerning this structure as well, we will deal with at another time.

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1 Introduction

Many results on semigroups based on ideals, bi-ideals and quasi-ideals are due to S. Lajos. In his paper in [2] he proved that if S is a regular semigroup, then a nonempty subset B of S is a bi-ideal of S if and only if there exists a right ideal R and a left ideal L of S such that $B = RL$. J. Calais proved in [1] that a semigroup S is regular if and only if the right and the left ideals of S are idempotent and for every right ideal A and every left ideal B of S , the product AB is a quasi-ideal of S . In the present note we present analogous results for hypersemigroups. We tried to use sets instead of elements in an attempt to show that, exactly as in semigroups, for the results on hypersemigroups based on ideals, points do not play any essential role, but the sets which shows their pointless character.

An *hypergroupoid* is a nonempty set H with an hyperoperation

$$\circ : H \times H \rightarrow \mathcal{P}^*(H) \mid (a, b) \rightarrow a \circ b$$

on H and an operation

$$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \rightarrow \mathcal{P}^*(H) \mid (A, B) \rightarrow A * B$$

on $\mathcal{P}^*(H)$ (induced by the operation of H) such that

$$A * B = \bigcup_{(a,b) \in A \times B} (a \circ b)$$

for every $A, B \in \mathcal{P}^*(H)$ ($\mathcal{P}^*(H)$ denotes the set of nonempty subsets of H).

An hypergroupoid $(H, \circ, *)$ is called *hypersemigroup* if

$$\{x\} * (y \circ z) = (x \circ y) * \{z\}$$

for all $x, y, z \in H$. Since $\{x\} * \{y\} = x \circ y$ for every $x, y \in H$, the associativity of H can be also defined by

$$\{x\} * (\{y\} * \{z\}) = (\{x\} * \{y\}) * \{z\}$$

for every $x, y, z \in H$.

A nonempty subset A of an hypergroupoid H is called a *left* (resp. *right*) ideal of H if $H * A \subseteq A$ (resp. $A * H \subseteq A$). A subset of H which is both a left and a right ideal of H is called an *ideal* of H . For any nonempty subset A of an hypersemigroup H , we denote by $R(A)$, $L(A)$ and $I(A)$ the right ideal, left ideal and the ideal of H , respectively, generated by A and we have

$$R(A) = A \cup (A * H), \quad L(A) = A \cup (H * A), \quad \text{and}$$

$$I(A) = A \cup (H * A) \cup (A * H) \cup (H * A * H).$$

A nonempty subset A of an hypergroupoid H is called *idempotent* if $A = A * A$.

2 Main results

By the definition of the hypergroupoid we have the following lemma which, though clear, plays an essential role in the investigation:

Lemma 1. *If H is an hypergroupoid and $A, B \in \mathcal{P}^*(H)$, then*

- (1) $x \in A * B \iff x \in a \circ b$ for some $a \in A, b \in B$.
- (2) If $a \in A$ and $b \in B$, then $a \circ b \subseteq A * B$.

Lemma 2. *If H is an hypergroupoid then, for any $A, B, C, D \in \mathcal{P}^*(H)$, we have*

- (1) $A \subseteq B$ and $C \subseteq D \implies A * C \subseteq B * D$.
- (2) $A \subseteq B \implies A * C \subseteq B * C$ and $C * A \subseteq C * B$.
- (3) $H * H \subseteq H$.
- (4) $H * A \subseteq H$ and $A * H \subseteq H$.

Proof. (1) Let $A \subseteq B$ and $C \subseteq D$. If $x \in A * C$, then $x \in a \circ b$ for some $a \in A$, $b \in C$, then $x \in a \circ b$ for some $a \in B$, $b \in D$, so $x \in B * D$. The proof of the rest is similar. \square

Lemma 3. *If H be an hypergroupoid and $A_i, B \in \mathcal{P}^*(H)$, $i \in I$, then*

$$\left(\bigcup_{i \in I} A_i\right) * B = \bigcup_{i \in I} (A_i * B).$$

Proof. Let $x \in \left(\bigcup_{i \in I} A_i\right) * B$. Then $x \in a \circ b$ for some $a \in \left(\bigcup_{i \in I} A_i\right)$, $b \in B$. Since $a \in A_j$ for some $j \in I$ and $b \in B$, we have $a \circ b \subseteq A_j * B \subseteq \bigcup_{i \in I} (A_i * B)$. Let now $x \in A_j * B$ for some $j \in I$. Then $x \in a \circ b$ for some $a \in A_j$, $b \in B$. Since $a \in A_j \subseteq \bigcup_{i \in I} A_i$ and $b \in B$, we have $a \circ b \subseteq \left(\bigcup_{i \in I} A_i\right) * B$. Then we get $x \in \left(\bigcup_{i \in I} A_i\right) * B$. \square

Lemma 4. *If H is an hypersemigroup, then the operation “ $*$ ” on $\mathcal{P}^*(H)$ is associative, that is $(A * B) * C = A * (B * C)$ for any $A, B, C \in \mathcal{P}^*(H)$.*

According to this lemma, instead of $(A * B) * C$ or $A * (B * C)$, we write $A * B * C$. The proof of this lemma is due to M. Tsingelis and it is based on Lemma 3.

Definition 5. An hypersemigroup H is called *regular* if, for any $A \in \mathcal{P}^*(H)$, we have $A \subseteq A * H * A$.

Definition 6. Let H be an hypersemigroup. A nonempty subset B of H is called a *bi-ideal* of H if

$$B * H * B \subseteq B.$$

Proposition 7. *Let H be an hypersemigroup. If C is a right ideal of H and $D \in \mathcal{P}^*(H)$ (or D a left ideal of H and $C \in \mathcal{P}^*(H)$), then the set $B = C * D$ is a bi-ideal of H .*

Proof. Let C be a right ideal of H and D a nonempty subset of H such that $B = C * D$. Then $B * H * B \subseteq B$. Indeed: We have

$$\begin{aligned} B * H * B &= (C * D) * H * (C * D) = C * (D * H * C) * D \\ &\subseteq (C * H) * D \subseteq C * D = B. \end{aligned}$$

□

Theorem 8. *If H is a regular hypersemigroup then, for every bi-ideal B of H there exists a right ideal C and a left ideal D of H such that*

$$B = C * D.$$

Proof. Let B be a bi-ideal of H . Then $B * H * B \subseteq B$. Since H is regular, we have $B \subseteq B * H * B$, thus we have $B = B * H * B$. On the other hand, we have

$$\begin{aligned} R(B) * L(B) &= \left(B \cup (B * H) \right) * \left(B \cup (H * B) \right) \\ &= (B * B) \cup \left((B * H) * B \right) \cup \left(B * (H * B) \right) \cup \left((B * H) * (H * B) \right) \\ &= (B * B) \cup (B * H * B) \cup \left(B * (H * H) * B \right). \end{aligned}$$

Since $H * H \subseteq H$, we have $B * (H * H) \subseteq B * H$ and $\left(B * (H * H) \right) * B \subseteq (B * H) * B$. Thus we have

$$R(B) * L(B) = (B * B) \cup (B * H * B) = (B * B) \cup B.$$

In addition, $B * B = (B * H * B) * B = \left(B * (H * B) \right) * B$. Since $H * B \subseteq H$, we get $B * (H * B) \subseteq B * H$ and $\left(B * (H * B) \right) * B \subseteq (B * H) * B$. Thus we obtain $B * B \subseteq B * H * B = B$, and $R(B) * L(B) = B$, where $R(B)$ is a right ideal and $L(B)$ is a left ideal of H . □

By Proposition 7 and Theorem 8, we have the following theorem

Theorem 9. *If H is a regular hypersemigroup then B is a bi-ideal of H if and only if there exists a right ideal C and a left ideal D of H such that $B = C * D$.*

Definition 10. If H is an hypergroupoid, a nonempty subset Q of H is called a *quasi-ideal* of H if

$$(Q * H) \cap (H * Q) \subseteq Q.$$

Lemma 11. *If H is an hypergroupoid, A a right ideal and B a left ideal of H , then $A \cap B \in \mathcal{P}^*(H)$.*

Proof. Take an element $a \in A$ and an element $b \in B$ ($A, B \neq \emptyset$). Since $\{a\} \subseteq A$ and $\{b\} \subseteq B$, we have $\{a\} * \{b\} \subseteq A * B$. Since $\{a\} * \{b\} = a \circ b$, we have $a \circ b \subseteq A * B \subseteq A * H \subseteq A$ and $a \circ b \subseteq A * B \subseteq H * B \subseteq B$, so $a \circ b \subseteq A \cap B$. Since $a \circ b \in \mathcal{P}^*(H)$, we have $A \cap B \in \mathcal{P}^*(H)$. \square

Theorem 12. *An hypersemigroup H is regular if and only if the right and the left ideals of H are idempotent, and for every right ideal A and every left ideal B of H , the product $A * B$ is a quasi-ideal of H .*

Proof. \implies . Let A be a right ideal of H . Since H is regular, we have

$$A \subseteq (A * H) * A \subseteq A * A \subseteq A * H \subseteq A,$$

thus we have $A * A = A$. Similarly, the left ideals of H are idempotent. Let now A be a right ideal and B a left ideal of H . Then $A \cap B$ is a quasi-ideal of H . In fact: Since $A \cap B \subseteq A, B$, we have $(A \cap B) * H \subseteq A * H \subseteq A$ and $H * (A \cap B) \subseteq H * B \subseteq B$, thus we have

$$\left((A \cap B) * H \right) \cap \left(H * (A \cap B) \right) \subseteq A \cap B \quad (*)$$

On the other hand, $A \cap B = A * B$. Indeed: Since $A \cap B \in \mathcal{P}^*(H)$ and H is regular, we have

$$A \cap B \subseteq (A \cap B) * H * (A \cap B) \subseteq A * H * B = (A * H) * B \subseteq A * B.$$

We also have $A * B \subseteq A * H \subseteq A$ and $A * B \subseteq H * B \subseteq B$, so $A * B \subseteq A \cap B$, then $A * B = A \cap B$. Thus the set $A * B$ is a quasi-ideal of H .

\Leftarrow . Let A be a nonempty subset of H . By hypothesis, we have

$$\begin{aligned} A &\subseteq R(A) = R(A) * R(A) = \left(A \cup (A * H) \right) * \left(A \cup (A * H) \right) \\ &= (A * A) \cup \left((A * H) * A \right) \cup \left(A * (A * H) \right) \cup \left((A * H) * (A * H) \right) \\ &= (A * A) \cup \left(A * (H * A) \right) \cup \left(A * (A * H) \right) \cup \left(A * (H * A) * H \right). \end{aligned}$$

Since $A \subseteq H$, we get $A * A \subseteq A * H$. Since $H * A \subseteq H$, we have $A * (H * A) \subseteq A * H$. Since $A * H \subseteq H$, $A * (A * H) \subseteq A * H$. Since $A * (H * A) \subseteq A * H$, we have

$A * (H * A) * H \subseteq A * (H * H) \subseteq A * H$. Thus we have $A \subseteq A * H$. Since $L(A)$ is a left ideal of H , in a similar way we get $A \subseteq H * A$. Therefore we obtain

$$A \subseteq (A * H) \cap (H * A).$$

Since $A * H$ is a right ideal and $H * A$ is a left ideal of H , by hypothesis, they are idempotent, and we have

$$\begin{aligned} (A * H) \cap (H * A) &= \left((A * H) * (A * H) \right) \cap \left((H * A) * (H * A) \right) \\ &= \left((A * H * A) * H \right) \cap \left(H * (A * H * A) \right). \end{aligned}$$

On the other hand, $A * H * A$ is a quasi-ideal of H . Indeed: Since H is a right ideal of H , by hypothesis it is idempotent that is, $H * H = H$. Thus we have

$$A * H * A = A * (H * H) * A = (A * H) * (H * A).$$

Since $A * H$ is a right ideal and $H * A$ a left ideal of H , by hypothesis, $(A * H) * (H * A)$ is a quasi-ideal of H . So $A * H * A$ is a quasi-ideal of H as well, which means that

$$\left((A * H * A) * H \right) \cap \left(H * (A * H * A) \right) \subseteq A * H * A.$$

Then we have $A \subseteq A * H * A$, and H is regular. \square

Corollary 13. [2] *If S is a regular semigroup, then B is a bi-ideal of S if and only if there exists a right ideal C and a left ideal D of S such that $B = CD$.*

Corollary 14. [1] *A semigroup S is regular if and only if the right and the left ideals of S are idempotent, and for every right ideal A and every left ideal B of S , the product AB is a quasi-ideal of S .*

Note. Although the corresponding results on semigroups can be also obtained as application of the results of this paper, exactly as in Γ -semigroups, we never work directly on hypersemigroups. We first have to prove the results on semigroups and then to transfer them to hypersemigroups. So although the results on hypersemigroups generalize the corresponding results on semigroups, they are completely based on the results on semigroups. We can do nothing on an hypersemigroup if we do not have it first for a semigroup. We wrote this paper in an attempt to show the way we pass from semigroups to hypersemigroups and give the right information concerning this structure. We will give further information concerning this structure in a next paper.

References

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